

# Microwave background constraints on inflationary parameters

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## ABSTRACT

We use a compilation of cosmic microwave anisotropy data (including the recent VSA, CBI and Archeops results), supplemented with an additional constraint on the expansion rate, to directly constrain the parameters of slow-roll inflation models. We find good agreement with other papers concerning the cosmological parameters, and display constraints on the power spectrum amplitude from inflation and the first two slow-roll parameters, finding in particular that  $\epsilon_1 < 0.057$ . The technique we use for parametrizing inflationary spectra may become essential once the data quality improves significantly.

**Key words:** cosmology: theory – cosmic microwave background.

## 1 INTRODUCTION

Recent measurements of the cosmic microwave background (CMB) show a flat portion at low multipole number  $\ell$  and a sharp peak around  $\ell \sim 200$ , as well as tentative evidence for a peak structure beyond  $\ell = 200$ . This represents a tremendous success for the simplest models of the universe described by a flat Friedmann–Robertson–Walker metric with adiabatic perturbations, which are in excellent qualitative agreement with these observations. The power of the CMB is that it can be used to constrain cosmological parameters, as well as allowing us to test our assumptions about the form of the initial irregularities in qualitative and now quantitative ways. The most popular assumption concerning the initial irregularities is that they originated during a period of cosmological inflation (see Liddle & Lyth 2000 for an extensive account of inflationary cosmology).

There have now been several papers which have searched for possible effects in this data from quite complicated inflationary models. One example is the inclusion of extra isocurvature degrees of freedom in the primordial power spectrum in addition to a dominant adiabatic component (Trotta, Riazuelo & Durrer 2001; Amendola et al. 2002), with the conclusion that the current data set is consistent with a subdominant isocurvature component (or even a dominant one on large scales in the case where the isocurvature perturbations are correlated with the adiabatic ones) and that the allowed values and ranges of the cosmological parameters are sensitive to the type of perturbations under consideration. Another example is attempts to fit inflation-motivated ‘features in the power spectrum’ to the data (Griffiths, Silk & Zaroubi 2001; Barriga et al. 2001; Adams, Cresswell & Easther 2001). These scalar power spectra have the intrinsic property of introducing extra degrees of freedom – the shape

parameters associated with the feature – which can be used alter the peak heights at will. However it is necessary to choose the features to coincide with characteristic scales in the CMB power spectrum, such as the first or second peaks [early work in this direction (Adams, Ross & Sarkar 1997) was also motivated by possible features in the matter power spectrum]. The CMB spectrum alone offers no evidence for any extra features and so smooth power spectra are currently best motivated.

It is surprising that relatively little attention has been paid to the CMB spectrum resulting from slow-roll inflation, even though these models have been the most intensively studied since its conception. Slow-roll inflation has acted as a guiding principle for inflation model builders, and is the simplest assumption and capable of giving excellent agreement with observations. The reason why specific studies of slow-roll inflation have been lacking is the usual assumption that inflation predicts a nearly power-law shaped spectrum, and hence that any information about inflation can be extracted as some linear combination of the constraints on the two key parameters, the scalar spectral index  $n_s$  and the tensor fraction  $R$ , an approach used by Kinney, Melchiorri & Riotto (2001) and by Hannestad et al. (2002) to discuss constraints on inflation. Hansen & Kunz (2002) also included the running of the spectral index, translating constraints on these parameters to place bounds on derivatives of the inflaton potential. Other parameter analyses (Wang, Tegmark & Zaldarriaga 2002; Percival et al. 2002) have tended to focus on results for other cosmological parameters such as the densities of the various matter components, and have been content to use this simple parametrization.

However, this approach ignores the fact that current data are only weakly constraining, and the current data set permits parameter regions where a significant deviation from a Harrison–Zel’dovich spectrum is allowed, and where the use of the full slow-roll power spectra is required to obtain robust results. The principal aim of this paper is to make the first direct estimation of slow-roll inflation parameters from CMB data. While the full predictions are relevant only in extreme regions of parameter space given current data, as the

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global cosmological data set improves (and in particular as significant observational weight develops on the high  $\ell$  part of the CMB spectrum) it may well be that these types of corrections take on increasing importance, depending which (if any) inflation models prove capable of fitting the data.

We do not expect any dramatic new results from this analysis given the quality of present data, though it is a useful test of the robustness of results under more general forms of the initial power spectra. However it is an important test of principle to bring these methods to bear on cosmological parameter estimation, as the high-quality data of coming years may well require the high-accuracy description of the power spectrum that this approach allows.

## 2 INFLATIONARY PARAMETERS

In order to implement the inflationary cosmology into a parameter search method, we need an adequate parametrization of the scalar and tensor power spectra that give rise to the observed anisotropies. We use the recently introduced horizon-flow parameters (Schwarz, Terrero-Escalante & García 2001), which are based around the Hubble parameter during inflation and its derivatives. These parameters enter directly into both the Friedmann equation and into the mode equations for scalar and tensor perturbations. They often simplify the results from analytical calculations of the power spectrum because of their simple definition

$$\epsilon_0 = H_{\text{inf}}/H; \quad (1)$$

$$\epsilon_{i+1} \equiv \frac{d \ln |\epsilon_i|}{dN}, \quad i \geq 0, \quad (2)$$

where  $H_{\text{inf}}$  is the Hubble parameter at some chosen time and  $N$  is the number of  $e$ -foldings of inflation. More importantly from the observational side, this formalism provides a consistent and detailed description of the shapes of the scalar and tensor power spectra as well as their absolute and relative normalizations, independent of other cosmological parameters (such as the cosmological constant physical density  $\omega_\Lambda$ ).

Our inflationary parameter set consists of the parameters  $\{H_{\text{inf}}, \epsilon_1, \epsilon_2, \epsilon_3, \dots\}$ , evaluated at some particular time during inflation. As the amplitude of inflationary perturbations is primarily determined around horizon crossing, we can relate time to scale by asking when a given scale equalled the Hubble radius during inflation,  $k = aH$ , and we can then think of these parameters as a function of scale. The scale at which they will be evaluated is in a sense arbitrary but is of course most wisely chosen to be around the centre of the scales actually probed by the observations. This parameter set replaces that made up of astrophysical quantities  $\{\mathcal{P}_{\mathcal{R}}, R_{10}(\Omega_\Lambda, h), n_S - 1, dn_S/d \ln k, n_T, dn_T/d \ln k, \dots\}$  that are used if the power spectra are taken as the starting point. Here  $\mathcal{P}_{\mathcal{R}}$  is the amplitude of scalar perturbations specified at the scale  $k = 0.05 \text{ Mpc}^{-1}$ ,  $R_{10}$  is the ratio of the tensor and scalar  $C_\ell$  curves evaluated at  $\ell = 10$ , and  $n_S - 1$  and  $n_T$  are the slopes of the initial scalar and tensor power-law spectra. Use of the inflationary parameters automatically enforces the inflationary consistency relations between scalar and tensor perturbations, and so there are fewer parameters to be fit; a full treatment of observations would also test these conditions though present data is not good enough for a meaningful confirmation.

The current data set has very limited abilities to constrain tensor perturbations and usually only  $R$  is used to describe them (perhaps coupled with a consistency relation to fix the tensor spectral index). It is also common, though not universal, for param-

eter searches to neglect scale dependence of the scalar spectral index.

The power spectrum  $\mathcal{P}$  from slow-roll inflation can be obtained as an expansion of the power spectrum (or some function of the power spectrum) in terms of the logarithmic wavenumber. While the usual power-law expression is mostly simply related to an expansion of  $\ln \mathcal{P}$ , since it is the power spectrum itself that is constrained by observations, it is most direct to expand  $\mathcal{P}(k)$  itself (Martin, Riazuelo & Schwarz 2000; Leach et al. 2002). The spectra of curvature perturbations  $\mathcal{P}_{\mathcal{R}}$  and of tensor perturbations  $\mathcal{P}_h$  are written as

$$\mathcal{P}_{\mathcal{R},h} \propto b_0 + b_1 \ln \left( \frac{k}{k_*} \right) + \frac{b_2}{2} \ln^2 \left( \frac{k}{k_*} \right) + \dots \quad (3)$$

The detailed predictions for the  $b_i$  have been calculated to second order in the slow-roll parameters (Stewart & Gong 2001; Leach et al. 2002) to be

$$\begin{aligned} b_{S0} = & 1 - 2(C + 1)\epsilon_1 - C\epsilon_2 + \left( 2C^2 + 2C + \frac{\pi^2}{2} - 5 \right) \epsilon_1^2 \\ & + \left( C^2 - C + \frac{7\pi^2}{12} - 7 \right) \epsilon_1 \epsilon_2 + \left( \frac{1}{2}C^2 + \frac{\pi^2}{8} - 1 \right) \epsilon_2^2 \\ & + \left( -\frac{1}{2}C^2 + \frac{\pi^2}{24} \right) \epsilon_2 \epsilon_3, \end{aligned} \quad (4)$$

$$b_{S1} = -2\epsilon_1 - \epsilon_2 + 2(2C + 1)\epsilon_1^2 + (2C - 1)\epsilon_1 \epsilon_2 + C\epsilon_2^2 - C\epsilon_2 \epsilon_3, \quad (5)$$

$$b_{S2} = 4\epsilon_1^2 + 2\epsilon_1 \epsilon_2 + \epsilon_2^2 - \epsilon_2 \epsilon_3 \quad (6)$$

for the scalars, and

$$\begin{aligned} b_{T0} = & 1 - 2(C + 1)\epsilon_1 + \left( 2C^2 + 2C + \frac{\pi^2}{2} - 5 \right) \epsilon_1^2 \\ & + \left( -C^2 - 2C + \frac{\pi^2}{12} - 2 \right) \epsilon_1 \epsilon_2, \end{aligned} \quad (7)$$

$$b_{T1} = -2\epsilon_1 + 2(2C + 1)\epsilon_1^2 - 2(C + 1)\epsilon_1 \epsilon_2, \quad (8)$$

$$b_{T2} = 4\epsilon_1^2 - 2\epsilon_1 \epsilon_2 \quad (9)$$

for the tensors, where  $C \equiv \gamma_E + \ln 2 - 2 \approx -0.7296$  is a numerical constant. The full parametrization of inflation as given here, where we truncate at  $\epsilon_3$ , should remain sufficiently accurate for some time to come, quite likely including *Planck* satellite results.

Concerning the relative normalization of tensor and scalars, the constant of proportionality for equation (3) is given by the slow-roll amplitudes

$$\mathcal{P}_{h0}(k_*) = \frac{16H_{\text{inf}}^2}{\pi m_{\text{Pl}}^2}, \quad (10)$$

$$\mathcal{P}_{\mathcal{R}0}(k_*) = \frac{H_{\text{inf}}^2}{\pi \epsilon_1 m_{\text{Pl}}^2}. \quad (11)$$

In the slow-roll limit the tensor to scalar ratio is given by  $\mathcal{P}_h/\mathcal{P}_{\mathcal{R}} = 16\epsilon_1$ , and the more general expression can be read from the above expansion coefficients as

$$\frac{\mathcal{P}_h}{\mathcal{P}_{\mathcal{R}}} = 16\epsilon_1[1 + C\epsilon_2 + \dots]. \quad (12)$$

This logarithmic expansion of the power spectra is accurate over a wide range of the inflationary parameter space. However, unlike the expansion of  $\ln \mathcal{P}(k)$  (the power-law expansion), the logarithmic

expansion of  $\mathcal{P}(k)$  can become negative at large  $|\ln(k/k_*)|$  for large  $\epsilon_i$ . This pathological behaviour serves as a warning against using either expansion alone for robustly extracting any inflationary signal in this regime without cross-checks. In any case, this behaviour does not occur in our analysis due to the limited range of scales currently probed by the CMB. The methods that we describe in this paper therefore complement the traditional approach, and a careful analysis should utilize both (Leach et al. 2002).

To lowest order in slow-roll these predictions reduce to the well-known linearized form

$$n_S - 1 = -2\epsilon_1 - \epsilon_2, \quad (13)$$

$$n_T = -2\epsilon_1, \quad (14)$$

$$R \equiv \frac{\mathcal{P}_h}{\mathcal{P}_R} = 16\epsilon_1. \quad (15)$$

Written in this way it is clear that an observational strategy would ideally use information from the tensor sector (primarily the tensor amplitude) to break the degeneracy between  $\epsilon_1$  and  $\epsilon_2$  for the scalar spectral index. The effect of increasing  $\epsilon_1$  is to boost the large-angle anisotropies via the tensor amplitude while simultaneously tilting downwards the small-angle anisotropies via the scalar tilt, which act roughly constructively.

In slow-roll inflation it is possible to relate the horizon-flow parameters to the shape of the inflationary potential

$$H^2 \simeq \frac{8\pi}{3m_{\text{Pl}}^2} V, \quad (16)$$

$$\epsilon_1 \simeq \frac{m_{\text{Pl}}^2}{16\pi} \left( \frac{V'}{V} \right)^2, \quad (17)$$

$$\epsilon_2 \simeq \frac{m_{\text{Pl}}^2}{4\pi} \left[ \left( \frac{V'}{V} \right)^2 - \frac{V''}{V} \right], \quad (18)$$

$$\epsilon_2 \epsilon_3 \simeq \frac{m_{\text{Pl}}^4}{32\pi^2} \left[ \frac{V'''V'}{V^2} - 3 \frac{V''}{V} \left( \frac{V'}{V} \right)^2 + 2 \left( \frac{V'}{V} \right)^4 \right]. \quad (19)$$

### 3 OBSERVATIONAL CONSTRAINTS

We fit to a CMB data set comprising data from *COBE*, *BOOMERanG*, *Maxima*, Degree Angular Scale Interferometer (DASI), Very Small Array (VSA), Cosmic Background Imager (CBI) and Archeops. We follow the method of Lesgourgues & Liddle (2001) in defining a simple  $\chi^2$  with penalty functions over  $D_\ell \equiv (\Delta T_\ell)^2 = \ell(\ell+1)C_\ell/2\pi$  for the *COBE* (Bennett et al. 1996; Tegmark & Hamilton 1997), *BOOMERanG* (Netterfield et al. 2002), *Maxima* (Lee et al. 2001) and Archeops (Benoit et al. 2003) data of the form

$$\chi^2 = \sum_\ell \frac{[D_\ell^{\text{theo}} - (1 + c\sigma_c + b\sigma_{b,1})D_\ell^{\text{obs}}]^2}{\sigma_\ell^2} + b^2 + c^2, \quad (20)$$

where for *COBE* we have  $\sigma_{b,1} = \sigma_c = 0$ , *BOOMERanG*  $\sigma_c = 0.20$  and  $\sigma_{b,1} = 0.43 \times 10^{-6}\ell^2$ , *Maxima*  $\sigma_{b,1} = 2 \times 10^{-6}\ell^{1.7}$  and  $\sigma_c = 0.08$ , and Archeops  $\sigma_c = 0.14$  and  $\sigma_{b,1} = 0$ . For the DASI (Halverson et al. 2002), VSA (Scott et al. 2002) and CBI (Pearson et al. 2002) ('mosaic' configuration, odd binning up to  $\ell = 1500$ ) data we define the  $\chi^2$  to be

$$\chi^2 = [D_i^{\text{theo}} - (1 + c\sigma_c)D_i^{\text{obs}}] M_{ij}^{-1} [D_j^{\text{theo}} - (1 + c\sigma_c)D_j^{\text{obs}}] + c^2, \quad (21)$$

where  $\sigma_c = 0.08, 0.06, 0.10$  respectively and the correlation matrix is given by

$$M_{ij} = \Delta D_i V_{ij} \Delta D_j + \sigma_c^2 D_i D_j. \quad (22)$$

We assume Gaussian window functions, except for the CBI data for which the published window functions are used, and the Archeops data for which the window functions are taken to be top-hat functions. All error bars are approximated as symmetric, taking the larger error bar when they are not. We analytically determine the coefficients  $b_{\text{boom}}, c_{\text{boom}}, b_{\text{max}}, c_{\text{max}}, c_{\text{dasi}}$  (simply by simultaneously solving equations such as  $\partial\chi^2/\partial b_{\text{boom}} = 0$ ) and then sum the  $\chi^2$  over all the experiments.

The CMB models were calculated using the 2002 January version of CAMB (Lewis, Challinor & Lasenby 2000) which can be easily modified to incorporate absolute and relative normalizations of the scalar and tensor power spectra.<sup>1</sup> We note in passing that indeed the correct amplitude in the tensors is more important than the correct tensor tilt. Even when we move the pivot scale away from *COBE* scales where the tensor spectrum is physically relevant, the tensor to scalar ratio only runs weakly back to the *COBE* scales

$$\frac{\mathcal{P}_h}{\mathcal{P}_R} = 16\epsilon_1 [1 + (\ln(k/k_*) + C)\epsilon_2 + \dots]. \quad (23)$$

The actual shape of the tensor spectrum will have very little effect at this stage, and so we are certainly justified in taking the same pivot scale for both the scalar and tensor spectra.

We examine a set of spatially flat cosmologies using the parameters  $\{\omega_B, \omega_M, \omega_\Lambda, e^{-2\tau}, \mathcal{P}_R, \epsilon_1, \epsilon_2\}$  where  $\omega_i \equiv \Omega_i h^2$  measures the physical matter density and  $\tau$  is the optical depth to the last-scattering surface. Our grid runs over the range  $0.006 < \omega_B < 0.045$ ,  $0.03 < \omega_M < 0.28$ ,  $0.0 < \omega_\Lambda < 1.2$ ,  $0.28 < e^{-2\tau} < 1.0$ ,  $7 < \mathcal{P}_R < 10^{10} < 48$ ,  $0.0001 < \epsilon_1 < 0.07$ ,  $-0.4 < \epsilon_2 < 0.3$ , with a uniform spacing of  $8^3 \times 7 \times 22 \times 8 \times 11$ . It is worth emphasizing that we have included  $\mathcal{P}_R$  as a full parameter instead of using an analytic procedure to choose the best amplitude from the data and each model. Apart from our obvious interest in primordial physics,  $\mathcal{P}_R e^{-2\tau}$  is a parameter that will be determined with high accuracy in the future and so should be included in the analysis.

The dependence of the power spectrum on the third slow-roll parameter  $\epsilon_3$  is very weak, given the current data set, and so we fix  $\epsilon_3 = 0$  while still using the second-order expressions, which give a better representation of the power spectrum at large  $\epsilon_1, \epsilon_2$  and  $|\ln(k/k_*)|$  than the first-order expressions. Our decision to truncate the slow-roll expansion after  $\epsilon_2$  is based on subtle considerations. As the series expansion is an infinite one, it clearly has to be truncated somewhere; if we did include  $\epsilon_3$  the same question of whether to include or not would then arise for  $\epsilon_4$ . We believe that a suitable criterion is that one ought not to include parameters which the data cannot distinguish from zero, i.e. we impose a null hypothesis that those parameters are zero. While that criterion is driven by data alone, it makes sense in the context of inflation because one does expect the parameters to be small, and if the data is unable to constrain them to be small, then the parameter space explored would include models which would not be satisfactory inflation models. Unfortunately this criterion cannot be rigidly applied, as even  $\epsilon_1$  and  $\epsilon_2$  cannot presently be distinguished from zero; thirty years of large-scale structure studies have not excluded the Harrison-Zel'dovich

<sup>1</sup> A module to directly input the predictions of slow-roll inflation to the CAMB program is available to download at [www.astronomy.sussex.ac.uk/~sleach/inflation/](http://www.astronomy.sussex.ac.uk/~sleach/inflation/).

spectrum. However if inflation is to be meaningfully tested, eventually at least one of these parameters needs to be detected with a non-zero value and so we do include them in our current treatment. This is appropriate as inflation models do predict values of  $\epsilon_1$  and  $\epsilon_2$  comparable to constraints from present data. By contrast, typical inflation models predict  $\epsilon_3$  to be much smaller than the present limits from data (see e.g. Hansen & Kunz 2001). Whether experiments such as *Planck* can detect a non-zero  $\epsilon_3$  remains to be seen.

We fix the pivot point to  $k_* = 0.05 \text{ Mpc}^{-1}$ , ignoring the cosmological dependence, proportional to  $\omega_M^{1/2}$ , of the range of scales probed by the CMB. The Hubble parameter is an auxiliary parameter given by

$$h = \sqrt{\omega_M + \omega_\Lambda}. \quad (24)$$

Throughout this paper we examine the effect of applying the *HST* Hubble parameter prior of  $h = 0.72 \pm 0.08$  at  $1\sigma$ , which acts as a constraint on the total physical matter density for our flat cosmologies.

When presenting parameter constraints, we simply minimize the  $\chi^2$  over unwanted parameters. In principle one should consider performing a cubic spline over the  $\chi^2$  of our coarse grid of models in order to examine the shape of the likelihood function  $e^{-\chi^2/2}$  and subsequently to perform a marginalization procedure (Efstathiou et al. 1999). However, minimizing the  $\chi^2$  reproduces the basic shape of the allowed region in parameter space (Tegmark & Zaldarriaga 2000). Our best-fitting model has  $\chi^2 = 51.1$  and so we plot the  $2\sigma$  and  $3\sigma$  contours for a  $\chi^2$  distribution with 51.1 degrees of freedom. We omit the  $1\sigma$  contour, as little useful information is conveyed by this level of certainty in the context of CMB parameter searches. We regard the models enclosed by the  $2\sigma$  contours as representing good fits to the data, and the  $3\sigma$  contours mark out the region where tension between the data and the models begins.

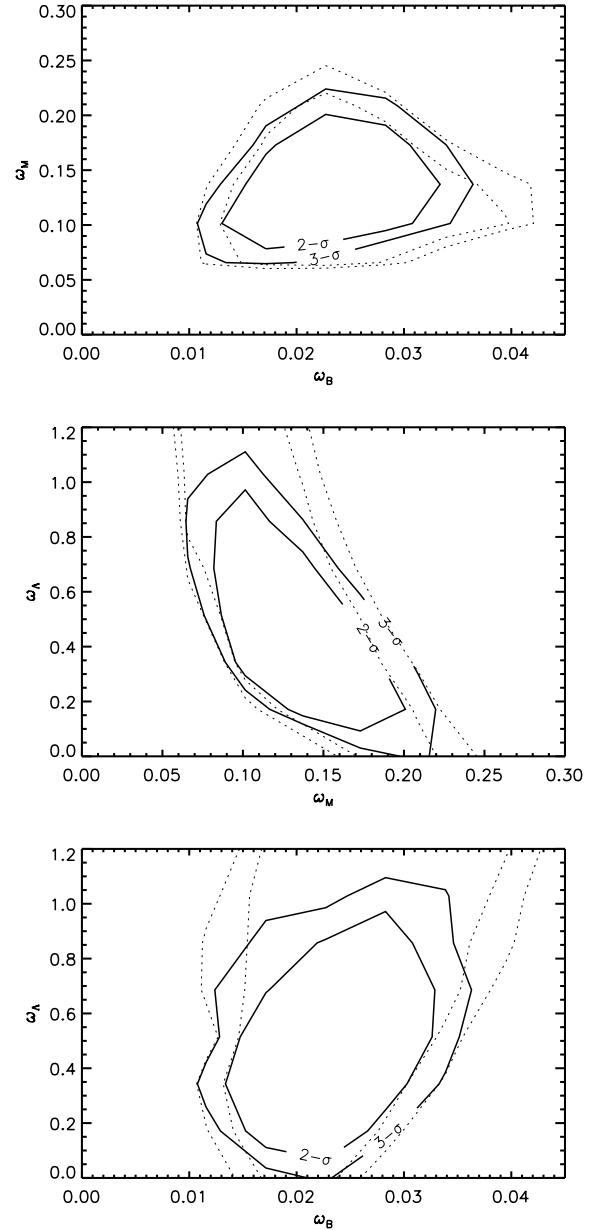
Turning first to three of the standard cosmological parameters, we see in Fig. 1 impressive agreement with the standard BBN value of  $\omega_B = 0.020 \pm 0.002$  at 95 per cent confidence (Burles, Nollett & Turner 2001). This acts as a useful consistency check on the assumption of adiabaticity, given that the inclusion of a subdominant isocurvature mode tends to widen the allowed range in  $\omega_B$  (Trotta et al. 2001). For the current data set we observe a weak correlation between the constraints on  $\omega_M$  and  $\omega_\Lambda$ . The main effect of applying a prior on the Hubble parameter,  $h$ , is to rule out models with a large physical density in  $\omega_\Lambda$ , and to close the contours near  $\omega_\Lambda = 0$ . When quoting parameter constraints we include the *HST* prior.

We now turn to the inflationary parameters, which are the main focus of our study. In Fig. 2 we plot the constraints on  $\mathcal{P}_\mathcal{R}$  and  $\epsilon_1$ . The general trend is that as we increase the tensor component,  $R = 16\epsilon_1$ , which contributes at low  $\ell$ , then we must decrease the scalar normalization in order to continue to fit the low  $\ell$  data. Including  $\mathcal{P}_\mathcal{R}$  as a full parameter in the analysis acts as a useful warning against including higher-order parameters in the power spectrum; until we have pinned down the amplitude of perturbations at some given scale, there is little sense in trying to measure the higher derivatives of the power spectrum such as the running of the spectral index etc, under the assumption that these higher derivatives are weak. Reading off the  $2\sigma$  bounds on  $\mathcal{P}_\mathcal{R}$  and  $\epsilon_1$  we find

$$11 < \mathcal{P}_\mathcal{R} \times 10^{10} < 42, \quad (25)$$

$$\epsilon_1 < 0.057. \quad (26)$$

These are the main results of this paper, and are in good agreement with the analysis of Lewis & Bridle (2002). The upper bound on  $\epsilon_1$



**Figure 1.** Three views of the constraints on  $\omega_B$ ,  $\omega_M$  and  $\omega_\Lambda$ . The dashed lines show the constraints from the CMB alone, while the solid lines include the *HST* prior on the Hubble constant.

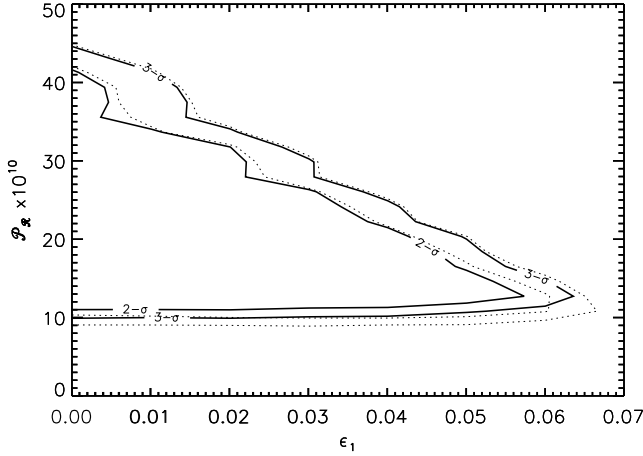
is consistent with the inflationary hypothesis, which requires  $\epsilon_1 < 1$  for inflation to occur. It is not possible to push this limit much further down using CMB data from ground and balloon observations, owing to the calibration uncertainties in the high  $\ell$  region that mask the primary effect of  $\epsilon_1$ , which is to boost the low  $C_\ell$ s relative to the high  $C_\ell$ s.

Reading off the values of  $\mathcal{P}_\mathcal{R}$  and  $\epsilon_1$  at the tip of the  $2\sigma$  contours of Fig. 2 allows us to place a constraint on the energy scale of inflation,

$$\frac{H_{\text{inf}}}{m_{\text{Pl}}} < 1.5 \times 10^{-5}, \quad (27)$$

$$V_{\text{inf}}^{1/4} < 2.9 \times 10^{16} \text{ GeV}. \quad (28)$$

Knox & Song (2002) and Kesden, Cooray & Kamionkowski (2002) have calculated a limit on  $V_{\text{inf}}^{1/4}$  below which tensors can not be



**Figure 2.** Constraints on the amplitude  $\mathcal{P}_{\mathcal{R}}$  and the first slow-roll parameter  $\epsilon_1$ , the solid lines again including the *HST* prior.

detected directly in the *B*-mode polarization of the CMB due to a contaminating signal from lensing of the *E*-mode along the line of sight. Combining this with our upper limit then we find that the inflationary energy scale would have to lie in the range

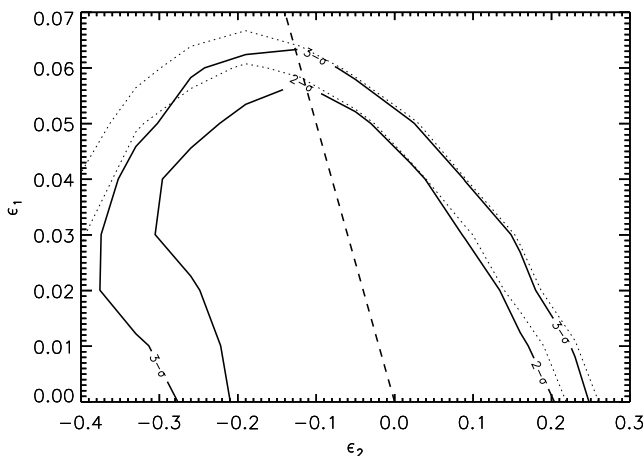
$$3 < \frac{V_{\text{inf}}^{1/4}}{10^{15} \text{ GeV}} < 29 \quad (29)$$

in order to detect directly the tensor spectrum via the *B*-mode polarization of the CMB.

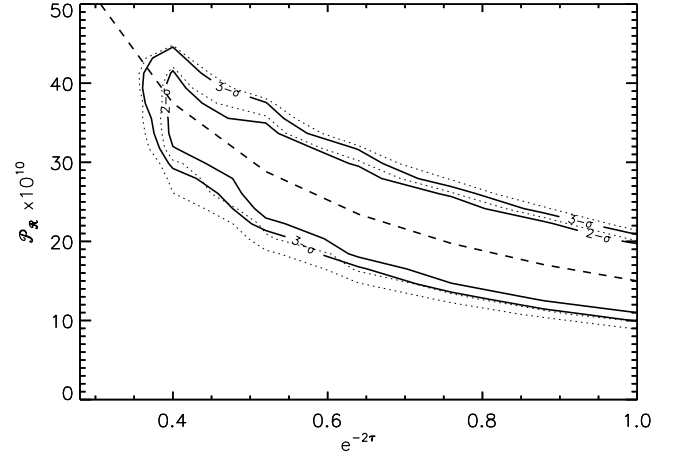
In Fig. 3 we plot the constraints on  $\epsilon_1$  and  $\epsilon_2$ , the first two slow-roll parameters, showing that a wide range of slow-roll inflation models fit the present data. The data are consistent with a scale-invariant scalar spectrum with no tensors ( $\epsilon_1 \ll 1$ ,  $|\epsilon_2| \ll 1$ ). We can also read off an approximate and loose bound on the second slow-roll parameter

$$-0.31 < \epsilon_2 < 0.2. \quad (30)$$

To guide the eye we plot the line  $\epsilon_1 = -\epsilon_2/2$  along which the inflationary scalar power spectrum is approximately scale-invariant. As can be expected, the contours lean in the same direction as this line reflecting the main inflationary degeneracy of equation (13).



**Figure 3.** Constraints on  $\epsilon_1$  and  $\epsilon_2$ . The data are consistent with a scale-invariant, tensorless spectrum ( $\epsilon_1 \ll 1$ ,  $|\epsilon_2| \ll 1$ ), but a significant tensor fraction and tilt are still permitted by the data. The dashed line,  $\epsilon_1 = -\epsilon_2/2$ , indicates inflationary models with a scale-invariant power spectrum. To the right the spectra are red, to the left they are blue.



**Figure 4.** Constraints on  $\mathcal{P}_{\mathcal{R}}$  and the optical depth  $\tau$ . The thick dashed curve is given by  $10^{10} \mathcal{P}_{\mathcal{R}} e^{-2\tau} = 15$ .

Finally, in Fig. 4 we plot the constraints on  $\mathcal{P}_{\mathcal{R}}$  and optical depth  $\tau$  revealing the anticipated degeneracy between these two parameters, both of which affect the amplitude of the acoustic peaks. We derive the limit  $0.4 < e^{-2\tau} (< 1.0)$  corresponding to an upper limit  $\tau < 0.45$ .

A final comment about the relationship between the optical depth and the tensor spectrum is useful. On large scales the amplitude of CMB anisotropies is sensitive to

$$\ell(\ell+1)C_\ell \propto \mathcal{P}_{\mathcal{R}} + c\mathcal{P}_h = \mathcal{P}_{\mathcal{R}}(1 + cR), \quad (31)$$

where the parameter  $c$  encodes the transfer of the primordial fluctuations to the CMB fluctuations on large scales, and has some cosmology dependence, in particular the late-time integrated Sachs–Wolfe effect. On small scales the tensor spectrum decays away, but the effect of reionization now becomes important, and the overall amplitude of CMB anisotropies is sensitive to

$$\ell(\ell+1)C_\ell \propto \mathcal{P}_{\mathcal{R}} e^{-2\tau}. \quad (32)$$

Thus, using data from both large- and small-scale CMB observations will allow us to probe the ratio of equations (31) and (32), an observable that we label as

$$\mathcal{A}_{\text{CMB}} \propto (1 + cR)e^{2\tau} \simeq 1 + cR + 2\tau, \quad (33)$$

where the final approximation holds in the late reionization, subdominant tensor limit. Equation (33) expresses the fact that that interplay between the tensor and reionization sectors is possible, leading to approximately the same observed spectrum. Physically speaking, this is because both mechanisms have the same effect, which is to decrease the ratio of small to large angular scale power. In any case, the upper limit on the tensor fraction,  $R$ , can always be determined by setting  $\tau = 0$ , and hence is independent of the range of  $\tau$  that we have and that is typically considered in CMB parameter searches. If a lower limit on  $\tau$  is determined by some other method then this enables us to push the upper limit on the tensor fraction down further.

#### 4 SUMMARY

We have implemented the detailed second-order predictions for the inflationary power spectra, given by equations (4) to (11), into a CMB parameter search method, using the logarithmic expansion of  $\mathcal{P}(k)$ , equation (3), for the first time. Although the present data set



cannot hope to actually measure the weak running of the spectral index induced by a significant tensor component, we derived sensible limits on the power spectrum amplitude  $\mathcal{P}_{\mathcal{R}}(k = 0.05 \text{ Mpc}^{-1})$  and the first two parameters,  $\epsilon_1$  and  $\epsilon_2$ , by assuming the running to be weak, which was achieved by considering models with  $\epsilon_1 < 0.07$ ,  $-0.4 < \epsilon_2 < 0.3$  (and fixing  $\epsilon_3 = 0$ ). We also derived a sensible limit for  $\omega_B$  which acts as a useful consistency check on the assumption of adiabatic perturbations with an approximately power-law form.

While the results of the present paper do not add much to existing studies parametrizing the spectra as power-laws, our paper represents an important point of principle in implementing precise slow-roll inflation predictions for the first time. As the global data set improves, including MAP and then *Planck* data, it is quite likely that these techniques are required to ensure robust estimation even of cosmological parameters such as the densities of the various components. Further, these techniques will be essential to squeeze the maximum possible amount of information out of the data regarding inflation, should slow-roll inflation continue to give the simplest viable interpretation of observational data. As the data set improves, it will be interesting to open up the  $\epsilon_3$  direction as well as exploring the possibility of a negligible tensor prior, in order to differentiate between inflationary models as effectively as possible. An interesting goal would be to determine the signs of both  $\epsilon_2$  and  $\epsilon_3$ , which would allow us immediately to rule out three quarters of all single-field slow-roll inflation models.

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